Math 8 Homework 4

1 Orderings and Other Relations

(a) An irreflexive relation R is one for which xRx is never true. Give an example of such a relation.

(b) An antisymmetric relation R is one for which xRy and yRx implies $x = y$. Describe all equivalence relations which are also antisymmetric.

(c) A partial ordering is a relation which is reflexive, antisymmetric, and transitive. Given a partial ordering $\leq$, we can define $\geq$ via the (obvious) rule $x \leq y$ if and only if $y \geq x$. Prove that $\geq$ is also a partial ordering.

(d) A total ordering is a relation which is partial ordering with the additional assumption that given any two x, y in the underlying set, $x \leq y$ or $y \leq x$. Give an example of a partial ordering which is not a total ordering.

(e) Given a partial ordering $\leq$ on a set S and an object $z \notin S$, we can extend $\leq$ to $S \cup \{z\}$ with the rule $x \leq z$ for all $x \in S$. Prove that this extension is still a partial ordering.

(f) Let L be the set of all lines in the plane and $\perp$ be defined by $\ell_1 \perp \ell_2$ if and only if ℓ_1 is perpendicular to ℓ_2. Is $\perp$ transitive? Symmetric? Antisymmetric? Reflexive?

2 Equivalence Relations

(a) Prove the following are equivalence relations (these are all important examples)

- (i) On $\mathbb{Z}$ define $x \equiv y$ if and only if there is some $k \in \mathbb{Z}$ so that $x - y = 7k$.
- (ii) On $\mathbb{R}$ define $x \simeq y$ if and only if $x - y \in \mathbb{Z}$.
- (iii) On $[0,1] \times [0,1]$ define $(x,y) \sim (w,z)$ if and only if either $(x,y) = (w,z)$ or both $x = w$ and $y + z = 1$.
- (iv) On the square $S = [0,1] \times [0,1]$ define its boundary $\partial S = (\{0,1\} \times [0,1]) \cup ([0,1] \times \{0,1\})$. Define $(x,y) \approx (w,z)$ if and only if $(x,y) = (w,z)$ or both $(x,y), (w,z) \in \partial S$.

(b) For each of the above equivalence relations, describe the collection of equivalence classes. Most of them have a geometric meaning; try to include this 'pictorial' interpretation in your description.

(c) The following is a false proof that transitivity and symmetry implies reflexivity. Find the flaw.

Proof. From $x \sim y$, symmetry implies $y \sim x$. Transitivity lets us combine these into $x \sim x$. $\square$

(d) Let S be a nonempty set. Find all equivalence relations $R \subseteq S \times S$ which are also functions (using the formal definition of a function as a set of ordered pairs).

(e) Let $C^1(\mathbb{R})$ be the set of functions $\mathbb{R} \to \mathbb{R}$ with continuous derivatives. Define $f \sim g$ to mean that $f' = g'$ everywhere. Prove there exists a bijection $C^1(\mathbb{R})/\!\sim\, \to \mathbb{R}$.

3 Constructing the Rational Numbers

You need not take new mathematical objects on blind faith. For example, why can we just declare -1 has a square root? Let's take a look at fractions and how they're rigorously defined. If we didn't "believe in" $\mathbb{Q}$ before, the steps below show how to build something that works just like rationals should. Furthermore, this construction can be used on more general algebraic objects; in abstract lingo, we've localized the ring $\mathbb{Z}$.

(a) Define $\simeq$ on $\mathbb{Z} \times (\mathbb{Z} - \{0\})$ as $(a,b) \simeq (x,y)$ if and only if $ay = bx$. Prove that $\simeq$ is an equivalence relation.

(b) Consider the set of equivalence classes $(\mathbb{Z} \times (\mathbb{Z} - \{0\}))/\simeq$, which we will rename Q for brevity. We'll also abbrieviate the equivalence classes; $[a,b]$ represents the equivalence class of (a,b). We can define addition in Q by the rule $[a,b] + [x,y] = [ay + bx, by]$. Prove this is "well–defined" in the following sense: if $(a,b) \simeq (c,d)$ and $(x,y) \simeq (z,w)$, then $(ay + bx, by) \simeq (cw + dz, dw)$.

(c) Prove the map $f : Q \to \mathbb{Q}$ given by $f([a,b]) = a/b$ is a well-defined bijection and that $f([a,b] + [x,y]) = f([a,b]) + f([x,y])$. Such an f is called an isomorphism, meaning that Q and $\mathbb{Q}$ look the same, as far as additive structure goes. The set Q is our construction of the rationals.